Properties of the Volume Corrected Characteristic Mixed Method

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Goal

Accurate, long-time simulation of transport processes.

Miscible, Incompressible Transport

Conservation of bulk fluid mass

$$\nabla\cdot\mathbf{u} = q$$

Conservation of component mass (ignore diffusion)

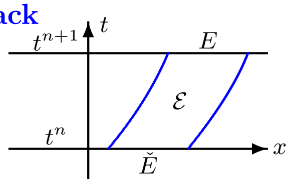
$$\phi c_t + \nabla\cdot(c\mathbf{u}) = q_c(c)$$

$\mathbf{u}$ is bulk fluid velocity — q are wells

c is component concentration — ϕ is porosity

Characteristic Trace-back

Fluid particles travel along characteristics. Particles in grid element E trace back to $\check{E}$.



Local Mass Conservation

The local mass constraint

$$\int_E \phi c^{n+1}\,dx = \int_{\check{E}} \phi c^n\,dx + \iint_{\mathcal{E}} q_c\,dx\,dt$$

The local volume constraint ($c = 1$ above)

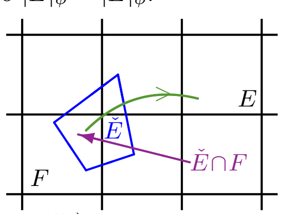
$$\int_E \phi\,dx = \int_{\check{E}} \phi\,dx + \iint_{\mathcal{E}} q\,dx\,dt$$

Volume Corrected Characteristics Mixed Method (VCCMM) in Algebraic Form

Pore volume in set S is $|S|_\phi$. Up to treatment of wells:

1. Trace back the corners of each grid element E.
2. Adjust traceback points so $|\check{E}|_\phi = |E|_\phi$.
3. Define the transport matrix indexed by the grid elements as

$$\mathbf{A}_{EF} = \frac{|\check{E}\cap F|_\phi}{|E|_\phi}$$



4. Transport c ($\mathbf{b}$ comes from wells) by

$$\mathbf{c}_E^{n+1} = \sum_F \mathbf{A}_{EF}\mathbf{c}_F^n + \mathbf{b}_E$$

5. (Optional) Postprocess c_h^{n+1} into a local linear $\tilde{c}_h^{n+1}$.

Letting $\mathbf{c}_h$ be the vector of element concentrations, this is

$$\mathbf{c}_h^{n+1} = \mathbf{A}\mathbf{c}_h^n + \mathbf{b} \quad \text{(local mass constraint)}$$

Algebraic Properties

Ignoring wells:

Nonnegativity $\quad \mathbf{A}_{EF} \ge 0$

Row sum $\quad \sum_F \mathbf{A}_{EF} = 1 \quad$ (local volume constraint)

Weighted column sum

$$\sum_E |E|_\phi \mathbf{A}_{EF} = |F|_\phi \quad \text{(tessellation constraint)}$$

Monotonicity and Maximum Principles

For two transport problems, if initially one concentration is below the other, then it stays below for all time:

$$c_{1,h}^0 \le c_{2,h}^0 \implies c_{1,h}^n \le c_{2,h}^n \quad \text{for all } n \ge 0.$$

There are no overshoots nor undershoots:

$$0 \le c_h^0 \le 1 \quad \text{for all } n \ge 0.$$

Stability

VCCMM is stable: if $\boldsymbol{\xi}_h^n$ satisfies the perturbed scheme

$$\boldsymbol{\xi}_h^{n+1} = \mathbf{A}\boldsymbol{\xi}_h^n + \mathbf{b} + \Delta t^n \boldsymbol{\delta}^n$$

where $\boldsymbol{\delta}^n$ is a perturbation, then to time T,

$$\max_{0\le n\le N} |\boldsymbol{\xi}_h^n - \mathbf{c}_h^n|_\infty \le |\boldsymbol{\xi}_h^0 - \mathbf{c}_h^0|_\infty + T \max_{0\le n\le N} |\boldsymbol{\delta}^n|_\infty.$$

Convergence

The convergence rate is optimal.

With physical diffusion

$$\max_{0\le n\le N} \|\tilde{c}_h^n - c^n\|_{L^2} \le C(h^2 + \Delta t),$$

$$\max_{0\le n\le N} \|c_h^n - c^n\|_{L^2} \le C(h + \Delta t).$$

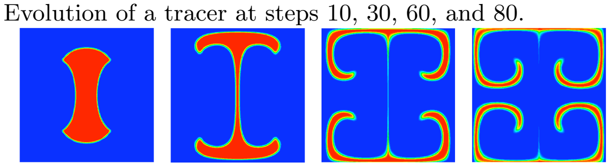
Without physical diffusion

$$\max_{0\le n\le N} \|c_h^n - c^n\|_{L^1} \le \|c_h^0 - c^0\|_{L^1} + C\left(\frac{h}{\sqrt{\Delta t}} + h + (\Delta t)^r\right),$$

where r is the accuracy of the traceback tracing. There is no restriction on Δt, and the results improve for large Δt.

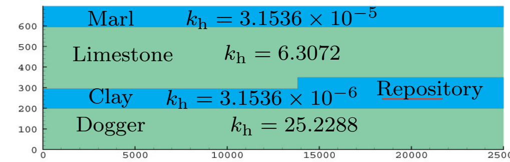
Example: Pure Curl Flow

Evolution of a tracer at steps 10, 30, 60, and 80.

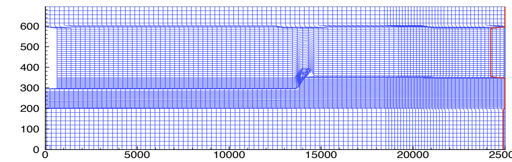


Extremely little numerical diffusion is seen!

Example: A Nuclear Waste Repository

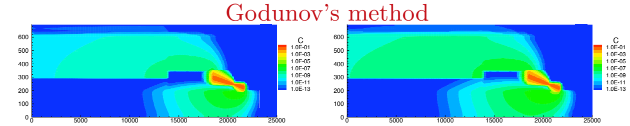


Characteristic trace-back mesh.

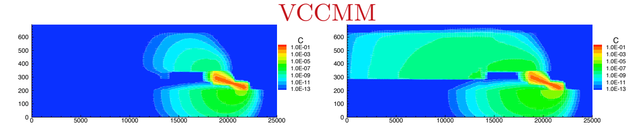


Concentration at 2.5×10^5 and 3×10^5 years.

Godunov's method



VCCMM



Computation time (sec).

	flow	trace & adjust	transport
Godunov	0.05	N/A	66.77
VCCMM	0.05	2.97	39.77

Conclusions

1. VCCMM is stable and accurate, even for large Δt;
2. has no overshoots nor undershoots;
3. is locally conservative for component *and* bulk fluid;
4. exhibits very little numerical diffusion;
5. can be more efficient for long-time simulations.

References

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